Kuwait University
Faculty of Science
Department of Mathematics

Math 101: Calculus I
Final Exam, August 7, 2011
Duration: 120 minutes

Calculators and Mobile Phones are not allowed.

1. (4 Points) Evaluate the following limits, if they exist.

   (a) $\lim_{x\to 0} \frac{\sin(x+1)}{x+1}$.

   (b) $\lim_{x\to 0} \left[(2x+1) + x^4 \cos\left(\frac{1}{x^4}\right)\right]$.

2. (4 Points) Find the x-coordinates of the points on the graph $y = 3x^{\frac{4}{3}} + 3x^{\frac{1}{3}} + 1$ where:

   (a) the tangent line is horizontal

   (b) the tangent line is vertical.

3. (4 Points) Let

$$f(x) = \begin{cases} 1 & \text{if } x < 0, \\ 3 & \text{if } x \geq 0. \end{cases}$$

   Is f differentiable at $x = 0$? (Justify your answer).

4. (4 Points) Let A be the area of a square, and P be its perimeter. Find the minimum value of $y = A - P$.

5. (4 Points) Let

$$f(x) = \tan(x).$$

   (a) Determine which conditions of Rolle's Theorem are satisfied on the interval $[0, \pi]$.

   (b) Show that there is no $c \in (0, \pi)$ such that $f'(c) = 0$? Does this contradict Rolle's Theorem? (Justify your answer).

6. (4 Points) Show that the function

$$f(x) = \int_0^{1+x^3} (1 + \cos^2(\sqrt{t}))\, dt + \int_0^{8\pi} (t^2 \sin^3(t) + 1)\, dt$$

   is increasing on $\mathbb{R}$.

7. (4 Points) Find the area of the region enclosed between the curves $y = x^2 - 5x$ and $y = x$.

8. (6 Points) Evaluate

   (a) $\int 4x(x^2+1)^5\, dx$.

   (b) $\int_{-2}^{2} \left(\sqrt{4-x^2} + \sin(x^3)\right)\, dx$.

9. (6 Points) Set up an integral for the volume that is obtained by revolving the region enclosed between the curves $y = x^2 - 5x$ and $y = x$ about the lines:

   (a) $x = -1$

   (b) $y = 7$.

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1. (4 Points) Evaluate the following limits, if they exist.

   (a) $\lim_{x\to 0} \frac{\sin(x+1)}{x+1} = \sin(1)$.

   (b) $\lim_{x\to 0} \left[(2x+1) + x^4 \cos\left(\frac{1}{x^4}\right)\right] = 1$.

   Because $\lim_{x\to 0} [(2x+1)] = 1$ and $\lim_{x\to 0} \left[x^4 \cos\left(\frac{1}{x^4}\right)\right] = 0$ (by the Sandwich Theorem).

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2. (4 Points) Find the x-coordinates of the points on the graph $y = 3x^{\frac{4}{3}} + 3x^{\frac{1}{3}} + 1$ where:

   (a) the tangent line is horizontal

   (b) the tangent line is vertical.

$$y' = 4x^{\frac{1}{3}} + x^{\frac{-2}{3}} = \frac{4x+1}{x^{\frac{2}{3}}}.$$

   a) $y' = 0$ when $x = \frac{-1}{4}$. $\therefore$ the tangent line is horizontal at $x = \frac{-1}{4}$.

   b) Since $\lim_{x\to 0} |\frac{4x+1}{x^{\frac{2}{3}}}| = \infty$ and the function $f(x)$ is continuous at $x = 0$ $\therefore$ the tangent line is vertical at $x = 0$.

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3. (4 Points) Let

$$f(x) = \begin{cases} 1 & \text{if} \quad x < 0, \\ 3 & \text{if} \quad x \geq 0. \end{cases}$$

   Is f differentiable at $x = 0$? (Justify your answer).

   Since $\lim_{x\to 0^-} f(x) = 1$ and $\lim_{x\to 0^+} f(x) = 3$ $\therefore$ f is discontinuous at $x = 0$, hence f is not differentiable there.

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4. (4 Points) Let A be the area of a square, and P be its perimeter. Find the minimum value of $y = A - P$.
   The area $A = x \cdot x = x^2$ and $P = 4x$. The expression $y = A - P = x^2 - 4x$. Thus, $y' = 2x - 4$: $y' = 0$ when $x = 2$. Since $y' < 0$ when $x < 2$ and $y' > 0$ when $x > 2$ $\therefore$ there is a local minimum (which is also a global minimum) at $x = 2$. The minimum value is $y = 2^2 - 4(2) = -4$.

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5. (4 Points) Let

$$f(x) = \tan(x).$$

   (a) Determine which conditions of Rolle's Theorem are satisfied on the interval $[0, \pi]$.
   $f(0) = 0$ and $f(\pi) = 0$.
   f is discontinuous at $x = \frac{\pi}{2}$, and hence f is not differentiable there.

   (b) Show that there is no $c \in (0, \pi)$ such that $f'(c) = 0$? Does this contradict Rolle's Theorem? (Justify your answer).

   $f'(x) = \sec^2(x) \neq 0$ for all $x \in (0, \pi)$. This does not contradict Rolle's Theorem since the hypotheses of the theorem are not satisfied i.e., f is discontinuous at $\frac{\pi}{2}$ (and not differentiable there).
